

QUANTIZATION OF COMPLEX CONTACT MANIFOLDS

PIETRO POLESELLO

ABSTRACT. A (holomorphic) quantization of a complex contact manifold is given by a filtered stack which is locally equivalent to the ring \mathcal{E} of microdifferential operators and which has trivial graded. The existence of a canonical quantization has been proved by Kashiwara. In this paper we first consider the classification problem, showing that these quantizations are classified by means of a certain sheaf homogeneous forms. Secondly, we consider the problem of existence and classification for quantization algebras.

INTRODUCTION

This paper deals with the problem of existence and classification of the quantizations of a complex contact manifold. Precisely, consider a complex contact manifold (Y, α) , that is, a complex manifold Y endowed with a projective 1-form α (an holomorphic section of the projective cotangent bundle $\pi: P^*Y \rightarrow Y$) such that $d\alpha|_H$ is non-degenerate, where H denotes the codimension 1 subbundle of the tangent bundle TY associated to α . Let $\mathcal{L} = \alpha^* \mathcal{O}_{P^*Y}(1)$ be the pull-back of the Serre sheaf on P^*Y : it is a locally free \mathcal{O}_Y -module of rank one endowed with the Lie bracket defined by α (the Lagrange bracket). Finally, denote by \mathcal{S}_Y the sheaf on Y whose local sections are symbols, that is, series $\sum_{j=-\infty}^m f_j$ with $f_j \in \mathcal{L}^{\otimes j}$, satisfying some growth conditions. This is endowed with a natural filtration and the associated graded sheaf is the "homogeneous coordinates" ring $\mathcal{O}_Y^{hom} = \bigoplus_{m \in \mathbb{Z}} \mathcal{L}^{\otimes m}$.

A (holomorphic) quantization algebra on (Y, α) is a sheaf of filtered algebras \mathcal{A} on Y which has \mathcal{O}_Y^{hom} as graded sheaf and which is locally

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isomorphic to \mathcal{S}_Y as filtered \mathbb{C} -modules, in such a way that the product \star on \mathcal{S}_Y induced by that of \mathcal{A} is given by bidifferential operators and is compatible both with the algebra structure on \mathcal{O}_Y and the Lie bracket on \mathcal{L} (that is to say, the algebra isomorphism $\mathcal{G}r(\mathcal{A}) \simeq \mathcal{O}_Y^{hom}$ preserves the graduation and the Poisson structures, the latter being induced by the commutator and by the Lie bracket, respectively).

By Darboux's theorem for contact manifolds, any point $p \in Y$ has an open neighborhood V and a contact transformation $i: V \rightarrow P^*M$, for a complex manifold M (here P^*M is endowed with the canonical contact structure given by the Liouville 1-form). Moreover, any quantization algebra on V is locally isomorphic through i to the \mathbb{C} -algebra \mathcal{E}_M of Sato's microdifferential operators. It follows that the quantization algebras on Y are nothing but \mathcal{E} -algebras, *i.e.*, \mathbb{C} -algebras locally isomorphic to $i^{-1}\mathcal{E}_M$ for any Darboux chart $i: Y \supset U \rightarrow P^*M$.

The \mathcal{E} -algebras on $Y = P^*M$ were classified by Boutet de Monvel [3]. In the general case, the situation is more complicated, since these objects may not exist globally. However Kashiwara [18] proved that there exists a canonical stack (sheaf of categories) of modules over locally defined \mathcal{E} -algebras. In fact, this stack is equivalent to the stack of modules over an algebroid stack \mathbf{E}_Y (see [21, 9]). Moreover, \mathbf{E}_Y has the same properties of an \mathcal{E} -algebra: it is filtered, locally equivalent to an \mathcal{E} -algebra and it has \mathcal{O}_Y^{hom} as associated graded stack. Thus, it makes sense to say that \mathbf{E}_Y is a (holomorphic) quantization of Y and to define a (\mathcal{E}, σ) -algebroid as a filtered \mathbb{C} -linear stack which is locally equivalent to \mathbf{E}_Y and which has \mathcal{O}_Y^{hom} as associated graded stack.

In this paper we classify these objects on any complex contact manifold Y by means of the cohomology group $H^1(Y; \Omega_Y^{1,cl}(0))$, where $\Omega_Y^{1,cl}(0)$ denotes the push-forward of the sheaf of closed 0-homogeneous 1-forms on the canonical symplectification of Y . Furthermore, we go into the study of the existence and classification of \mathcal{E} -algebras on Y , thus generalizing the results in [3]. Finally, in the formal case we compare the classification of \mathcal{E} -algebras with that of (\mathcal{E}, σ) -algebroids.

Note that the problem of the quantization of the homogeneous coordinates sheaf have been considered in [21, 15] in the projective Poisson/symplectic case. On real contact manifolds or, more generally, conic Poisson manifolds of constant rank, quantization algebras have been considered by Boutet de Monvel under the name of Toeplitz algebras (see [2, 4]).

This paper is organized as follows: in Section 1 we recall the main definitions and properties of complex contact manifolds. In Section 2 we review the definition of quantization algebras. In Section 3 we recall the classification of \mathcal{E} -algebras on P^*M . In Section 4 we define the (\mathcal{E}, σ) -algebroids on a complex contact manifold and classify them (Theorem 4.5); we then make some explicit computations on compact Kähler manifolds. In Section 5 we investigate the existence of \mathcal{E} -algebras on complex contact manifolds and compare their classification with that of (\mathcal{E}, σ) -algebroids, focusing mainly on the formal case.

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Notations and conventions All the filtrations are over \mathbb{Z} , increasing and exhaustive. If \mathcal{A} is a sheaf of filtered algebras, we will denote by $\mathcal{G}r(\mathcal{A})$ its associated sheaf of graded algebras, and by $\mathcal{G}r_i(\mathcal{A})$ the subsheaf of homogeneous elements of degree i . We will use similar notations for morphisms.

If \mathcal{A} is a sheaf of algebras, we will denote by \mathcal{A}^\times the sheaf of groups of its invertible elements and, for each section $a \in \mathcal{A}^\times$, by $\text{ad}(a): \mathcal{A} \rightarrow \mathcal{A}$ the algebra isomorphism $b \mapsto aba^{-1}$.

If Y is a complex manifold, Ω_Y^i will denote the sheaf of holomorphic i -forms and \mathcal{O}_Y (resp. Θ_Y) that of holomorphic functions (resp. vector fields). We also set $\Omega_Y = \Omega_Y^{\dim Y}$, the canonical bundle of Y . For each section $\zeta \in \Theta_Y$, the Lie derivative (resp. the contraction) along ζ is denoted by L_ζ (resp. i_ζ).

1. COMPLEX CONTACT MANIFOLDS

We recall here the definition and some basic properties of complex contact manifolds. References are made to [27, Section 3.3] and [23, 1].

1.1. Complex contact structures. Let Y be a complex manifold and $\pi: P^*Y \rightarrow Y$ its projective cotangent bundle (here $P^*M = \dot{T}^*M/\mathbb{C}^\times$, where \dot{T}^*M denotes the cotangent bundle of M with the zero-section removed). Recall that a projective 1-form on Y is a global section of π . By projective duality, this corresponds to an hyperplane distribution, *i.e.* a codimension 1 sub-bundle of the tangent bundle TY .

Given a projective 1-form $\alpha: Y \rightarrow P^*Y$, consider the cartesian diagram

$$\begin{array}{ccc} U = Y \times_{P^*Y} U^*Y & \longrightarrow & U^*Y \\ \downarrow \gamma & & \downarrow \\ Y & \xrightarrow{\alpha} & P^*Y, \end{array}$$

where $U^*Y \rightarrow P^*Y$ is the tautological (or universal) line bundle. By composing the morphism $U \rightarrow U^*Y$ with the natural projection $U^*Y \rightarrow T^*Y$, we get an injective morphism of vector bundles (denoted by the same symbol) $\alpha: U \rightarrow T^*Y$. Hence, the line bundle $\gamma: U \rightarrow Y$ measures the failure to lift the projective 1-form α to a nowhere vanishing 1-form on Y . Moreover, the chain of vector bundle morphisms

$$U \xrightarrow{\Delta} U \times_Y U \xrightarrow{id_U \times \alpha} U \times_Y T^*Y \xrightarrow{\gamma^t} T^*U$$

(here $\gamma^t: U \times_Y T^*Y \rightarrow T^*U$ denotes the morphism induced by the transpose of the differential $d\gamma_v: T_v U \rightarrow T_{\gamma(v)} Y$ of γ at $v \in U$) defines a 1-form $\tilde{\alpha}$ on U which never vanishes outside the zero-section.

Let eu be the infinitesimal generator of the \mathbb{C}^\times -action on U ; then $L_{eu}\tilde{\alpha} = \tilde{\alpha}$ (that is, $\tilde{\alpha}$ is 1-homogeneous) and $i_{eu}\tilde{\alpha} = 0$.

Let \mathcal{H} denote the sheaf of sections of the hyperplane distribution $H \subset TY$ associated to α and $\mathcal{O}_{P^*Y}(-1)$ the sheaf of sections of the tautological line bundle. Let $\mathcal{O}_{P^*Y}(1)$ be its dual and set $\mathcal{L} = \alpha^*\mathcal{O}_{P^*Y}(1)$. Then $\mathcal{L}^{\otimes -1}$ is the sheaf of sections of $\gamma: U \rightarrow Y$ and we get dual exact sequences of locally free \mathcal{O}_Y -modules

$$(1.1) \quad 0 \longrightarrow \mathcal{H} \longrightarrow \Theta_Y \xrightarrow{t_\alpha} \mathcal{L} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{L}^{\otimes -1} \xrightarrow{\alpha} \Omega_Y^1 \longrightarrow \mathcal{H}^{\otimes -1} \longrightarrow 0.$$

The differential $d\alpha: \mathcal{L}^{\otimes -1} \rightarrow \Omega_Y^2$ is by definition the composition of α with the de Rham differential. Its restriction $d\alpha|_H$ is obtained by composing $d\alpha$ with $\Omega_Y^2 \rightarrow \bigwedge_{\mathcal{O}_X}^2 \mathcal{H}^{\otimes -1}$. One checks that the morphism thereby obtained is \mathcal{O}_X -linear, and so is the morphism $\alpha \wedge (d\alpha)^k: \mathcal{L}^{\otimes -(k+1)} \rightarrow \Omega_Y^{2k+1}$ for any $k \in \mathbb{N}$. This thus defines a global section of $\Omega_Y^{2k+1} \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes (k+1)}$.

Definition-Proposition 1.1. In the above situation, α is called a contact form if one of the following equivalent conditions is satisfied:

- i) The \mathcal{L} -valued skew form $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{H} \rightarrow \mathcal{L}$, $(v, w) \mapsto {}^t\alpha([v, w])$ is everywhere non-degenerate (here $[\cdot, \cdot]$ denotes the Lie bracket on Θ_Y).
- ii) For any $l \in \mathcal{L}^{\otimes -1}$, the 2-form $d\alpha|_H(l)$ is everywhere non-degenerate.
- iii) $\dim Y = 2n + 1$ and $\alpha \wedge (d\alpha)^n$ is a nowhere vanishing section of $\Omega_Y \otimes_{\mathcal{O}_Y} \mathcal{L}^{\otimes (n+1)}$.
- iv) $\tilde{\omega} = d\tilde{\alpha}$ is a symplectic form on $\tilde{Y} = U \setminus \{\text{zero-section}\}$.

Note that iii) implies that $\mathcal{L}^{\otimes -(n+1)} \simeq \Omega_Y$ and from the Cartan formula follows that $L_{eu}\tilde{\omega} = \tilde{\omega}$, that is, $\tilde{\omega}$ is 1-homogeneous.

- Definition 1.2.**
- i) A complex contact manifold is a complex manifold Y endowed with a contact form α . If α lifts to a 1-form, the contact manifold (Y, α) is called exact.
 - ii) A contact transformation $\varphi: (Y, \alpha) \rightarrow (X, \beta)$ between complex contact manifolds of the same dimension is a morphism of complex manifolds satisfying $\varphi^*(\beta) = \alpha$.

Given a complex contact manifold (Y, α) , we will refer to \mathcal{L} (resp. \mathcal{H}) as the contact line bundle (resp. distribution), and to the principal \mathbb{C}^\times -bundle $\gamma: \tilde{Y} \rightarrow Y$ together with the 2-form $\tilde{\omega}$ on \tilde{Y} , as the canonical symplectification. Note that (Y, α) is exact if and only if \mathcal{L} is trivial or, equivalently, if γ has a global section. Note also that any contact transformation $\varphi: Y \rightarrow X$ is a local isomorphism and lifts to a \mathbb{C}^\times -equivariant symplectic transformation $\tilde{\varphi}: \tilde{Y} \rightarrow \tilde{X}$.

Remark 1.3. We include in the definition all 1-dimensional complex manifolds as degenerate case. Indeed, if $\dim Y = 1$ there is an identification $Y = P^*Y$, so that Y has a canonical contact form (which is in fact unique, up to isomorphism), whose contact line bundle is the sheaf Θ_Y of vector fields and the contact distribution nothing but its zero-section. Note that Y is exact if and only if there exists a globally defined nowhere vanishing vector field (or, equivalently, a globally defined nowhere vanishing 1-form).

Example 1.4. Let M be an $n+1$ -dimensional complex manifold. Then P^*M is a contact manifold of dimension $2n + 1$, where the canonical contact form $\lambda: P^*M \rightarrow P^*(P^*M)$ is obtained by projectivizing the

Liouville form on T^*M . Given $y \in P^*M$, let H_y the associated hyperplane in T_yM ; then $\lambda(y) \in P^*(P^*M)$ is represented by the hyperplane $d\pi_y^{-1}(H_y) \subset T_y(P^*M)$, where $d\pi_y: T_y(P^*M) \rightarrow T_{\pi(y)}M$ is the differential at y of the projection $\pi: P^*M \rightarrow M$. The contact line bundle is then $\mathcal{O}_{P^*M}(1)$ and the canonical symplectification is \dot{T}^*M endowed with the standard symplectic structure.

Note that each contact manifold (Y, α) is a contact submanifold of (P^*Y, λ) . Indeed, $\alpha: Y \rightarrow P^*Y$ is an embedding and $\alpha^*(\lambda) = \alpha$ by definition of the Liouville form.

Recall that the Darboux's theorem for contact manifolds asserts that a local model for a contact manifold Y is an open subset of P^*M for a complex manifold M . More precisely, for any point $y \in Y$ there exist an open neighborhood V of y , a complex manifold M and a contact transformation $i: V \rightarrow P^*M$. We call the pair (V, i) a Darboux chart.

Example 1.5. Let $J^1N = T^*N \times \mathbb{C}$ be the 1-jet bundle of an n -dimensional complex manifold N . Let (t, τ) be the system of homogeneous symplectic coordinates on $T^*\mathbb{C}$. Then J^1N is identified with the open subset of $P^*(N \times \mathbb{C})$ defined by $\tau \neq 0$, hence it is endowed with the contact form induced by that of $P^*(N \times \mathbb{C})$. If λ denotes the Liouville form on T^*N , then $\beta = \rho^*(\lambda) + dt$ is a lift of such contact form, $\rho: J^1N \rightarrow T^*N$ being the projection.

More generally, if (X, ω) is a complex symplectic manifold with $[\omega] = 0$ in $H^2(X; \mathbb{C}_X)$, there exists a principal \mathbb{C} -bundle $\rho: Y \rightarrow X$ and a 1-form β on Y such that $d\beta = \rho^*(\omega)$ and $i_v(\beta) = 1$, for v the infinitesimal generator of the \mathbb{C} -action. Then β is the lift of a contact form on Y . Such exact contact manifold is called a contactification of (X, ω) .

Other examples of exact contact manifolds are the Iwasawa manifolds and the 1-dimensional complex tori.

Example 1.6. The odd-dimensional complex projective space \mathbb{P}^{2n+1} is a contact manifold, whose contact line bundle is $\mathcal{O}_{\mathbb{P}^{2n+1}}(2)$. Any 2-homogeneous symplectic form ω on \mathbb{C}^{2n+2} defines a contact form on \mathbb{P}^{2n+1} by pushing-down $\frac{1}{2}i_{eu}\omega$, for eu the Euler vector field. The canonical symplectification is thus $\mathbb{C}^{2n+2} \setminus \{0\}$ with symplectic form $d(\frac{1}{2}i_{eu}\omega) = \omega$.

In homogeneous coordinates $[x_0, \dots, x_{2n+1}]$, the symplectic form $\omega = \sum_{i=0}^n dx_{2i} \wedge dx_{2i+1}$ defines the contact form $\alpha = \frac{1}{2} \sum_{i=0}^n (x_{2i} dx_{2i+1} - x_{2i+1} dx_{2i})$.

1.2. Lagrange brackets and Poisson structures. Let (Y, α) be a complex contact manifold, with contact line bundle \mathcal{L} , contact distribution \mathcal{H} and canonical symplectification $\gamma: \tilde{Y} \rightarrow Y$.

Definition 1.7. $v \in \Theta_Y$ is a contact vector field if $[v, w] \in \mathcal{H}$ for every $w \in \mathcal{H}$. We denote by $\Theta_Y^c \subset \Theta_Y$ the subsheaf of contact vector fields.

By the Jacobi identity, it follows that Θ_Y^c is closed under the Lie bracket, so that it inherits a Lie algebra structure. Note that a vector field is a contact vector field if and only if its local flow consists in contact transformations. In particular, $H^0(Y; \Theta_Y^c)$ is the Lie algebra of the complex Lie group of contact transformations of Y and $H^1(Y; \Theta_Y^c)$ classifies the infinitesimal contact deformations of Y (see [22]).

Proposition 1.8. *The morphism ${}^t\alpha: \Theta_Y \rightarrow \mathcal{L}$ in (1.1) restricts to an isomorphism of \mathbb{C} -modules $\Theta_Y^c \xrightarrow{\sim} \mathcal{L}$.*

Proof. By the exact sequence (1.1), $v \in \Theta_Y$ is a contact vector field if and only if ${}^t\alpha([v, w]) = 0$ for every $w \in \mathcal{H}$. Since by Definition 1.1 (i) the skew form ${}^t\alpha([\cdot, \cdot])$ on \mathcal{H} is non-degenerate, a contact vector field v belongs to \mathcal{H} if and only if $v = 0$. Hence ${}^t\alpha$ is injective on Θ_Y^c .

Take $l \in \mathcal{L}$. Since ${}^t\alpha: \Theta_Y \rightarrow \mathcal{L}$ is surjective, locally there exists $u \in \Theta_Y$ such that ${}^t\alpha(u) = l$. Consider the map $\mathcal{H} \rightarrow \mathcal{L}$, $w \mapsto {}^t\alpha([u, w])$. It is \mathcal{O}_X -linear, hence there exists a unique section $u' \in \mathcal{H}$ such that ${}^t\alpha([u, \cdot]) = {}^t\alpha([u', \cdot])$ on \mathcal{H} . It follows that ${}^t\alpha([u - u', w]) = 0$ for every $w \in \mathcal{H}$, that is $u - u' \in \Theta_Y^c$, and ${}^t\alpha(u - u') = l$. \square

Denote by

$$R: \mathcal{L} \rightarrow \Theta_Y^c, \quad l \mapsto R_l$$

the inverse of ${}^t\alpha|_{\Theta_Y^c}$ (R_l is known as the Reeb field of l). This defines a \mathbb{C} -linear splitting of the exact sequence (1.1), therefore for each i there is a canonical splitting

$$H^i(Y; \Theta_Y) = H^i(Y; \mathcal{H}) \oplus H^i(Y; \mathcal{L}).$$

Let $l, m \in \mathcal{L}$ and set

$$\{l, m\} = {}^t\alpha([R_l, R_m]).$$

Then $\{\cdot, \cdot\}$ defines a Lie bracket on \mathcal{L} (the Lagrange bracket), and ${}^t\alpha$ and R thus become isomorphisms of sheaves of complex Lie algebras. Note that for any $f \in \mathcal{O}_X$ one has $\{l, fm\} = L_{R_l}(f)m + f\{l, m\}$.

Set

$$\mathcal{O}_Y^{hom} = \bigoplus_{m \in \mathbb{Z}} \mathcal{L}^{\otimes m}.$$

This is the graded algebra of "homogeneous coordinates", since there is an identification $\mathcal{O}_Y^{hom} = \gamma_* \mathcal{O}_{[\tilde{Y}]}$, where $\mathcal{O}_{[\tilde{Y}]} \subset \mathcal{O}_{\tilde{Y}}$ is the sheaf of holomorphic functions on \tilde{Y} which are rational on the fiber. It has a natural Poisson algebra structure defined by extending the Lagrange bracket $\{\cdot, \cdot\}$ on \mathcal{L} as a derivation on each argument. It is of degree -1 (i.e. $\deg\{l, m\} = \deg l + \deg m - 1$) and coincides with the Poisson structure induced by the bracket on \tilde{Y} associated to the symplectic form. In particular, one has $\{l, f\} = L_{R_l}(f)$ for any $l \in \mathcal{L}$ and $f \in \mathcal{O}_X$.

Notation 1.9. For any $k \in \mathbb{Z}$ we denote by $Der_{\mathbb{C}}^P(\mathcal{O}_Y^{hom})(k)$ the sheaf of \mathbb{C} -derivations of \mathcal{O}_Y^{hom} which are graded of degree k and preserve the Poisson structure¹.

Let $\Theta_Y^s(k)$ be the sheaf of k -homogeneous symplectic vector fields² on \tilde{Y} and $\Omega_Y^i(k)$ that of k -homogeneous, i -forms on \tilde{Y} , i.e. such that $L_{eu}(\omega) = k\omega$. Set

$$\Theta_Y^s(k) = \gamma_* \Theta_{\tilde{Y}}^s(k), \quad \Omega_Y^i(k) = \gamma_* \Omega_{\tilde{Y}}^i(k).$$

Then $\Theta_Y^s(0) = \Theta_Y^c$ and there are isomorphisms of \mathbb{C} -modules

$$L: \Theta_Y^s(k) \xrightarrow{\sim} Der_{\mathbb{C}}^P(\mathcal{O}_Y^{hom})(k), \quad H: \Omega_Y^{1,cl}(k) \xrightarrow{\sim} \Theta_Y^s(k-1)$$

induced by the Lie derivative and the Hamiltonian isomorphism on \tilde{Y} , respectively (here the upper index cl denotes closed forms). Hence we have an isomorphism of \mathbb{C} -modules

$$(1.2) \quad (L \circ H)^{-1}: Der_{\mathbb{C}}^P(\mathcal{O}_Y^{hom})(k-1) \xrightarrow{\sim} \Omega_Y^{1,cl}(k).$$

By identifying $\Omega_Y^0(k)$ with $\mathcal{L}^{\otimes k}$ and by using the Cartan formula, one gets for any $k \neq 0$ an isomorphism of \mathbb{C} -modules

$$d: \mathcal{L}^{\otimes k} \xrightarrow{\sim} \Omega_Y^{1,cl}(k).$$

¹Recall that a derivation $D: \mathcal{O}_Y^{hom} \rightarrow \mathcal{O}_Y^{hom}$ is graded of degree k if $D(\mathcal{L}^{\otimes m}) \subset \mathcal{L}^{\otimes m+k}$ for any $m \in \mathbb{Z}$, and it preserves the Poisson structure if $D(\{f, g\}) = \{D(f), g\} + \{f, D(g)\}$ for any $f, g \in \mathcal{O}_Y^{hom}$.

²Recall that a vector field v on $(\tilde{Y}, \tilde{\omega})$ is symplectic if $L_v \tilde{\omega} = 0$ and it is k -homogeneous if $L_{eu} v = [eu, v] = kv$.

In particular, for $k = 1$, one gets that the contact vector field R_l associated to $l \in \mathcal{L}$ is obtained by pushing down the symplectic vector field H_{dl} .

For $k = 0$, there is an exact sequence

$$0 \rightarrow \mathbb{C}_Y \rightarrow \mathcal{O}_Y \xrightarrow{d} \Omega_Y^{1,cl}(0) \xrightarrow{i_{eu}} \mathbb{C}_Y \rightarrow 0.$$

Together with the exact sequence

$$(1.3) \quad 0 \rightarrow \mathbb{C}_Y \rightarrow \mathcal{O}_Y \xrightarrow{d} \Omega_Y^{1,cl} \rightarrow 0,$$

it gives rise to the exact sequence

$$(1.4) \quad 0 \rightarrow \Omega_Y^{1,cl} \xrightarrow{\gamma^*} \Omega_Y^{1,cl}(0) \xrightarrow{i_{eu}} \mathbb{C}_Y \rightarrow 0$$

(here γ^* denotes the pull-back via $\gamma: \tilde{Y} \rightarrow Y$).

Lemma 1.10. *The class of the extension of \mathbb{C} -modules (1.4) is the Atiyah class³ $a(\mathcal{L}) \in H^1(Y; \Omega_Y^{1,cl})$ of the contact line bundle.*

Proof. The sequence (1.4) splits if and only if there exists a global section $\beta \in \Omega_Y^{1,cl}(0)$ such that $i_{eu}\beta = 1$, *i.e.* if and only if $d + \beta \wedge \cdot$ descends to a flat connection on \mathcal{L} , that is, to a global section of the $\Omega_Y^{1,cl}$ -torsor of the flat connections of \mathcal{L} . And this happens if and only if this torsor is trivial, hence if and only if its class $a(\mathcal{L})$ is 0 in $H^1(Y; \Omega_Y^{1,cl})$. \square

Note that the Atiyah class controls also the splitting of the exact sequence (1.1).

Remark 1.11. From (3.5) it follows that $H^1(Y; \Omega_Y^{1,cl})$ classifies TDO-rings on Y . Then $a(\mathcal{L})$ equals the class of the TDO-ring $\mathcal{D}_{\mathcal{L}} = \mathcal{L} \otimes_{\mathcal{O}_M} \mathcal{D}_M \otimes_{\mathcal{O}_M} \mathcal{L}^{\otimes -1} \simeq \mathcal{E}nd_{\mathcal{D}_M^{\text{op}}}(\mathcal{L} \otimes_{\mathcal{O}_M} \mathcal{D}_M)$ (see [17] for more details).

2. QUANTIZATION ALGEBRAS

We review here the definition of a quantization algebra (called star algebras in [3]).

Let (Y, α) be a complex contact manifold, $\gamma: \tilde{Y} \rightarrow Y$ its canonical symplectification, and \mathcal{L} the associated contact line bundle endowed with the Lagrange bracket $\{\cdot, \cdot\}$ defined by α .

³The standard Atiyah map $a: H^1(Y; \mathcal{O}_Y^\times) \rightarrow H^1(Y; \Omega_Y^1)$ induced by $d \log$ factors through $H^1(Y; \Omega_Y^{1,cl})$.

Let

$$F_m \widehat{\mathcal{S}}_Y = \prod_{j=-\infty}^m \mathcal{L}^{\otimes j}$$

be the sheaf of formal symbols of order $\leq m$, that is, series $\sum_{j=-\infty}^m f_j$ with $f_j \in \mathcal{L}^{\otimes j}$, and set

$$\widehat{\mathcal{S}}_Y = \bigcup_{m \in \mathbb{Z}} F_m \widehat{\mathcal{S}}_Y.$$

We denote by $\mathcal{S}_Y \subset \widehat{\mathcal{S}}_Y$ the subsheaf of (holomorphic) symbols, that is, symbols which, on an open subset $U \subset Y$ where they are defined, are subject to the estimates

$$(2.1) \quad \begin{cases} \text{for any compact subset } K \text{ of } \gamma^{-1}(U) \text{ there exists a constant} \\ C_K > 0 \text{ such that } \sup_K |f_j| \leq C_K^{-j} (-j)! \text{ for all } j < 0. \end{cases}$$

The sheaf \mathcal{S}_Y inherits the filtration from that of $\widehat{\mathcal{S}}_Y$, and one has

$$\mathcal{G}r(\mathcal{S}_Y) = \mathcal{G}r(\widehat{\mathcal{S}}_Y) = \mathcal{O}_Y^{hom}.$$

Remark 2.1. Since \mathcal{L} has rank 1, the algebra $F_0 \widehat{\mathcal{S}}_Y$ is nothing but the completion of the symmetric algebra of $\mathcal{L}^{\otimes -1}$. Therefore, it is identified with the sheaf $\mathcal{O}_{U^*} \hat{=} \hat{\mathcal{O}}_Y$, the formal completion of \mathcal{O}_{U^*} along Y , where U^* denotes the total space of \mathcal{L} and Y is identified with the zero section of $U^* \rightarrow Y$. Whence $F_0 \widehat{\mathcal{S}}_Y$ is the sheaf of holomorphic functions in the formal neighborhood of Y in U^* .

Recall that the Borel transform defines an isomorphism of \mathbb{C} -modules

$$B: \mathcal{O}_{U^*} \hat{=} \hat{\mathcal{O}}_Y \xrightarrow{\sim} \mathcal{O}_Y \hat{=} \hat{\mathcal{O}}_Y.$$

Via the previous identification, we get an isomorphism of \mathbb{C} -modules

$$L: F_0 \widehat{\mathcal{S}}_Y \xrightarrow{\sim} \mathcal{O}_Y \hat{=} \hat{\mathcal{O}}_Y.$$

In local coordinates $(y; \tau) \in U^*$ and $(y; t) \in U$, a section $f(y; \tau) \in F_0 \widehat{\mathcal{S}}_Y$ is written as $\sum_{j \leq 0} f_j(y) \tau^j$ with $f_j \in \mathcal{O}_Y$ and

$$L(f)(y; t) = \frac{1}{2\pi i} \int_{\gamma} f(y; \tau) \frac{e^{t\tau}}{\tau} d\tau = \sum_{j \geq 0} f_{-j}(y) \frac{t^j}{j!}$$

for γ a counter clockwise oriented circle around 0 in \mathbb{C} . Then L restricts to an isomorphism $F_0\mathcal{S}_Y \xrightarrow{\sim} \mathcal{O}_U|_Y$, since by (2.1) for any $K \subset \gamma^{-1}(U)$ compact the series $\sum_{j \geq 0} \sup_K |f_{-j}| \frac{C_K^j}{j!}$ converges.

An associative filtered \mathbb{C} -algebra law \star on \mathcal{S}_Y is differential if $\star = \sum_{j=-\infty}^0 B_j$ with B_j a bidifferential operator homogeneous of degree j , that is, a \mathbb{C} -linear morphism $B_j: \mathcal{L}^{\otimes m} \otimes_{\mathbb{C}} \mathcal{L}^{\otimes n} \rightarrow \mathcal{L}^{\otimes m+n+j}$ which is a differential operator⁴ in each argument.

Definition 2.2. A (holomorphic) quantization algebra on Y is a sheaf of filtered \mathbb{C} -algebras \mathcal{A} on Y which has \mathcal{O}_Y^{hom} as graded sheaf and which is locally isomorphic to \mathcal{S}_Y as filtered \mathbb{C} -modules, in such a way that the product \star on \mathcal{S}_Y induced by that of \mathcal{A} is differential and the following diagrams commute

$$(2.2) \quad \begin{array}{ccc} F_0\mathcal{A} \times F_0\mathcal{A} & \xrightarrow{\star} & F_0\mathcal{A} \\ \downarrow \nu_0 \times \nu_0 & & \downarrow \nu_0 \\ \mathcal{O}_Y \times \mathcal{O}_Y & \xrightarrow{\cdot} & \mathcal{O}_Y \end{array} \quad \begin{array}{ccc} F_1\mathcal{A} \times F_1\mathcal{A} & \xrightarrow{[\cdot, \cdot]} & F_1\mathcal{A} \\ \downarrow \nu_1 \times \nu_1 & & \downarrow \nu_1 \\ \mathcal{L} \times \mathcal{L} & \xrightarrow{\{\cdot, \cdot\}} & \mathcal{L} \end{array}$$

(here $[\cdot, \cdot]$ denotes the commutator and ν_i the symbol map of degree i).

Note that the diagrams (2.2) commute if and only if the algebra isomorphism $\mathcal{G}r(\mathcal{A}) \simeq \mathcal{O}_Y^{hom}$ preserves the graduations and the Poisson structures (where the Poisson structure on $\mathcal{G}r(\mathcal{A})$ is induced by $[\cdot, \cdot]$).

Remark 2.3. The same definition holds in the formal case, that is, replacing \mathcal{S}_Y by $\hat{\mathcal{S}}_Y$.

3. MICRODIFFERENTIAL ALGEBRAS ON P^*M

Since a local model for a contact manifold Y is an open subset of P^*M for a complex manifold M , we start here by considering the quantization algebras on P^*M , as in [3]. The sheaf of Sato's microdifferential operators provides the local model for such algebras.

⁴Recall that a \mathbb{C} -linear morphism $P: \mathcal{M} \rightarrow \mathcal{N}$ between \mathcal{O}_Y -modules is called a differential operator if for every $m \in \mathcal{M}$ there exist finitely many differential operators P_j and $n_j \in \mathcal{N}$ such that $P(gm) = \sum_j P_j(g)n_j$ for all $g \in \mathcal{O}_X$.

3.1. \mathcal{E} -algebras. Let M be an n -dimensional complex manifold and $\pi: P^*M \rightarrow M$ its projective cotangent bundle, endowed with the canonical contact structure given by the Liouville form. Recall from Example 4.12, that the contact line bundle is the relative Serre sheaf $\mathcal{O}_{P^*M}(1)$ and the canonical symplectification is the \mathbb{C}^\times -principal bundle $\gamma: \dot{T}^*M \rightarrow P^*M$.

Let \mathcal{E}_M be the sheaf on P^*M of microdifferential operators (see [27, 16, 19], and also [28] for an exposition). Recall that, in a local coordinate system (x) on M with associated local coordinates $(x; [\xi])$ on P^*M , a microdifferential operator P of order $\leq m$ has a total symbol

$$\sigma_{\text{tot}}(P) \in F_m \mathcal{S}_{P^*M},$$

and that the product structure is given by the Leibniz formula:

$$\sigma_{\text{tot}}(PQ) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_{\text{tot}}(P) \partial_x^\alpha \sigma_{\text{tot}}(Q)$$

for Q another microdifferential operator of total symbol $\sigma_{\text{tot}}(Q)$. With this product, \mathcal{E}_M is a filtered algebra with center the constant sheaf \mathbb{C}_{P^*M} .

Denote by $F_m \mathcal{E}_M$ the subsheaf of operators of order $\leq m$ and by

$$\sigma_m: F_m \mathcal{E}_M \rightarrow F_m \mathcal{E}_M / F_m \mathcal{E}_M(m-1) \simeq \mathcal{O}_{P^*M}(m)$$

the symbol map of order m . If $\sigma_m(P)$ is not identically zero, then one says that P has order m . In particular, $P \in F_m \mathcal{E}_M$ is invertible if and only if $\sigma_m(P)$ is nowhere vanishing.

The symbol maps induce an isomorphism of graded algebras

$$(3.1) \quad \sigma: \mathcal{G}r(\mathcal{E}_M) \xrightarrow{\sim} \mathcal{O}_{P^*M}^{\text{hom}}.$$

For any $P \in F_m \mathcal{E}_M$ and $Q \in F_n \mathcal{E}_M$ one has

$$\sigma_{m+n-1}[P, Q] = \{\sigma_m(P), \sigma_n(Q)\},$$

hence the morphism σ preserves the corresponding Poisson structures. It follows that \mathcal{E}_M is a quantization algebra on P^*M .

Lemma 3.1 (cf. [27]). *Let $\varphi: \mathcal{E}_M \rightarrow \mathcal{E}_M$ be a \mathbb{C} -algebra automorphism. Then φ is filtered with $\mathcal{G}r(\varphi) = \text{id}$.*

See [10, Lemma 4.2.1] for an elementary proof. This suggests the following:

Definition 3.2. An \mathcal{E} -algebra on P^*M is a \mathbb{C} -algebra locally isomorphic to \mathcal{E}_M as \mathbb{C} -algebras.

By Lemma 3.1, any \mathcal{E} -algebra \mathcal{A} comes equipped with a canonical filtration $\{F_m \mathcal{A}\}$ and with an isomorphism of graded algebras

$$\nu: \mathcal{G}r(\mathcal{A}) \xrightarrow{\sim} \mathcal{O}_{P^*M}^{hom}.$$

Also, any \mathbb{C} -linear isomorphism of \mathcal{E} -algebras $\varphi: \mathcal{A}_1 \rightarrow \mathcal{A}_2$ is filtered and compatible with the associated graded isomorphisms ν^1, ν^2 . This means that φ maps $F_m \mathcal{A}_1$ to $F_m \mathcal{A}_2$ in such a way that $\nu_m^2(\varphi(P)) = \nu_m^1(P)$ for all $P \in F_m \mathcal{A}_1$. (Here ν_m^i denotes the symbol map $F_m(\mathcal{A}_i) \rightarrow F_m(\mathcal{A}_i)/F_{m-1}(\mathcal{A}_i) \simeq \mathcal{O}_{P^*M}(m)$ of order m , for $i = 1, 2$.)

Given an \mathcal{E} -algebra \mathcal{A} , it follows that the associated graded isomorphism ν preserves the Poisson structures, so that \mathcal{A} is a quantization algebra. Moreover, the converse is also true:

Proposition 3.3 (cf. [3, Proposition 2]). *Any quantization algebra on P^*M is an \mathcal{E} -algebra.*

This allows us to identify quantization algebras with \mathcal{E} -algebras.

Example 3.4. Let \mathcal{P} be a locally free right \mathcal{E}_M -module of rank one. Then $\mathcal{E}nd_{\mathcal{E}_M^{\text{op}}}(\mathcal{P})$ is an \mathcal{E} -algebra. If $\mathcal{P} = \pi^{-1}\mathcal{N} \otimes_{\pi^{-1}\mathcal{O}_M} \mathcal{E}_M$ for a line bundle \mathcal{N} on M , the above \mathcal{E} -algebra may be written as

$$\mathcal{E}_{\mathcal{N}} = \pi^{-1}\mathcal{N} \otimes_{\pi^{-1}\mathcal{O}_M} \mathcal{E}_M \otimes_{\pi^{-1}\mathcal{O}_M} \pi^{-1}\mathcal{N}^{\otimes -1}.$$

In particular, $\mathcal{E}_{\Omega_M} \xleftarrow{\sim} \mathcal{E}_M^{\text{op}}$ by the formal adjoint (see [19]). More generally, we may replace \mathcal{N} by any \mathbb{C} -twisted line bundle (see [17, 8]).

Let $f: P^*M \rightarrow P^*N$ be a contact transformation. By a result of [27] (see also [16, 19]), there exists a $f^{-1}\mathcal{E}_N \otimes_{\mathbb{C}} \mathcal{E}_M$ -module \mathcal{M}_f which is locally free of rank one as \mathcal{E}_M -module. Then $\mathcal{P} = \pi^{-1}\Omega_M \otimes_{\pi^{-1}\mathcal{O}_M} \mathcal{M}_f$ is a $f^{-1}\mathcal{E}_N \otimes_{\mathbb{C}} \mathcal{E}_M^{\text{op}}$ -module locally free of rank one as right \mathcal{E}_M -module and $f^{-1}\mathcal{E}_N \xrightarrow{\sim} \mathcal{E}nd_{\mathcal{E}_M^{\text{op}}}(\mathcal{P})$. It follows that $f^{-1}\mathcal{E}_N$ is an \mathcal{E} -algebra.

Notation 3.5. We denote by

$$\{\mathcal{E}\text{-algebras}\}_{P^*M}$$

the set of isomorphism classes of \mathcal{E} -algebras on P^*M , pointed by the class of \mathcal{E}_M .

3.2. Classification. By definition of \mathcal{E} -algebra, it follows that there is an isomorphism of pointed sets

$$(3.2) \quad \{\mathcal{E}\text{-algebras}\}_{P^*M} \simeq H^1(P^*M; \text{Aut}_{\mathbb{C}\text{-alg}}(\mathcal{E}_M)),$$

where $\mathcal{A}ut_{\mathbb{C}\text{-alg}}(\mathcal{E}_M)$ denotes the sheaf of \mathbb{C} -algebra automorphisms of \mathcal{E}_M and the right-hand side is defined by Čech cohomology.

Let $\varphi: \mathcal{E}_M \rightarrow \mathcal{E}_M$ be a \mathbb{C} -algebra automorphism. By Lemma 3.1, it is filtered with $\mathcal{G}r(\varphi) = \text{id}$. Therefore the assignment

$$F_m \mathcal{E}_M \ni P \mapsto \sigma_{m-1}(\varphi(P) - P) \in \mathcal{O}_{P^*M}(m-1)$$

defines a \mathbb{C} -derivation of $\mathcal{O}_{P^*M}^{hom}$ graded of degree -1 . It is easily seen to preserve the Poisson structure, so that it is a section of $Der_{\mathbb{C}}^P(\mathcal{O}_{P^*M}^{hom})(-1)$.

Denote by $s(\varphi)$ the section of $\Omega_{P^*M}^{1,cl}(0)$ obtained by applying (1.2). Since $s(\varphi \circ \varphi') = s(\varphi) + s(\varphi')$, we get a group morphism

$$(3.3) \quad s: \mathcal{A}ut_{\mathbb{C}\text{-alg}}(\mathcal{E}_M) \rightarrow \Omega_{P^*M}^{1,cl}(0).$$

Note that, if $P \in F_0 \mathcal{E}_M^\times$, then $s(\text{ad}(P)) = d \log \sigma_0(P)$. Set

$$\mathcal{E}_{M,1} = \{P \in F_0 \mathcal{E}_M; \sigma_0(P) = 1\} \subset F_0 \mathcal{E}_M^\times.$$

Lemma 3.6 (cf. [3]). *The sequence of sheaves of groups on P^*M*

$$1 \rightarrow \mathcal{E}_{M,1} \xrightarrow{\text{ad}} \mathcal{A}ut_{\mathbb{C}\text{-alg}}(\mathcal{E}_M) \xrightarrow{s} \Omega_{P^*M}^{1,cl}(0) \rightarrow 0$$

is exact.

Consider the associated long non-abelian exact sequence

$$(3.4) \quad H^1(P^*M; \mathcal{E}_{M,1}) \rightarrow H^1(P^*M; \mathcal{A}ut_{\mathbb{C}\text{-alg}}(\mathcal{E}_M)) \xrightarrow{s} H^1(P^*M; \Omega_{P^*M}^{1,cl}(0)).$$

The pointed set $H^1(P^*M; \mathcal{E}_{M,1})$ classifies isomorphism classes of locally free right \mathcal{E}_M -module of rank one which are filtered with trivial graded (*i.e.*, isomorphic to $\mathcal{O}_{P^*M}^{hom}$) and the first map in the sequence is described by $[\mathcal{M}] \mapsto [\mathcal{E}nd_{\mathcal{E}^{\text{op}}}(\mathcal{M})]$ (here $[\cdot]$ denotes isomorphism class).

Let \mathcal{D}_M be the \mathbb{C} -algebra of linear differential operators on M . Recall that \mathcal{D}_M is identified to a filtered subalgebra of $\pi_* \mathcal{E}_M$ and that $\mathcal{G}r(\mathcal{D}_M) \simeq \bigoplus_{m \in \mathbb{N}} \pi_* \mathcal{O}_{P^*M}(m)$. Note that, if $\dim M \geq 2$, then for any $m < 0$ one has $\pi_* \mathcal{O}_{P^*M}(m) = 0$, hence $\mathcal{D}_M = \pi_* \mathcal{E}_M$.

Denote by $\mathcal{A}ut_{TDO}(\mathcal{D}_M)$ the group of automorphisms of \mathcal{D}_M as a TDO-ring (see [17], and also [8]). Then the assignment $\psi \mapsto \mathcal{G}r_1(\psi - \text{id})$ defines an isomorphism of groups

$$(3.5) \quad \mathcal{A}ut_{TDO}(\mathcal{D}_M) \xrightarrow{\sim} \Omega_M^{1,cl}.$$

In particular, TDO-ring automorphisms of \mathcal{D}_M are inner. Since $\mathcal{O}_M^\times = \mathcal{D}_M^\times \subset \pi_* \mathcal{E}_M^\times$, it follows that TDO-ring automorphisms of \mathcal{D}_M extend to inner automorphisms of \mathcal{E}_M . We thus get a commutative diagram

$$(3.6) \quad \begin{array}{ccc} \pi_* \mathcal{A}ut_{\mathbb{C}\text{-alg}}(\mathcal{E}_M) & \xrightarrow{s} & \pi_* \Omega_{P^*M}^{1,cl}(0), \\ \uparrow & & \uparrow \pi^* \\ \mathcal{A}ut_{TDO}(\mathcal{D}_M) & \xrightarrow{\sim} & \Omega_M^{1,cl} \end{array}$$

where π^* denotes the pull-back via $\pi: P^*M \rightarrow M$.

Proposition 3.7. *If $\dim M \geq 2$, then the \mathcal{E} -algebras on P^*M are classified by $H^1(P^*M; \mathcal{E}_{M,1}) \times H^1(M; \Omega_M^{1,cl})$.*

Proof. By using (3.6), one gets commutative diagrams

$$(3.7) \quad \begin{array}{ccc} H^i(P^*M; \mathcal{A}ut_{\mathcal{E}\text{-alg}}(\mathcal{E}_M)) & \xrightarrow{s} & H^i(P^*M; \Omega_{P^*M}^{1,cl}(0)), \\ \uparrow t & \nearrow \pi^* & \\ H^i(M; \Omega_M^{1,cl}) & & \end{array}$$

for $i = 0, 1$. If $\dim M \geq 2$, then π^* is an isomorphism (see [3]). It follows that (3.4) splits, and one then uses (3.2). \square

Recall from *loc. cit.* that $H^1(M; \mathcal{O}_M^\times / \mathbb{C}_M^\times) \simeq H^1(M; \Omega_M^{1,cl})$ classifies \mathbb{C} -twisted line bundles. The map t in (3.7) for $i = 1$ is thus given by

$$[\mathcal{N}] \mapsto [\mathcal{E}_{\mathcal{N}}]$$

(see Example 3.4). If $\dim M \geq 2$ then, up to isomorphism, any \mathcal{E} -algebras on P^*M is of the form $\pi^{-1}\mathcal{N} \otimes_{\pi^{-1}\mathcal{O}_M} \mathcal{E}nd_{\mathcal{E}^{\text{op}}}(\mathcal{P}) \otimes_{\pi^{-1}\mathcal{O}_M} \pi^{-1}\mathcal{N}^{\otimes -1}$ for \mathcal{P} a locally free right \mathcal{E}_M -module of rank one which is filtered with trivial graded and \mathcal{N} a \mathbb{C} -twisted line bundle on M .

Remark 3.8. All the results in this section hold in the formal case, that is, replacing \mathcal{E} -algebras by $\widehat{\mathcal{E}}$ -algebras, which are obtained by dropping the growth condition (2.1).

4. MICRODIFFERENTIAL ALGEBROIDS ON CONTACT MANIFOLDS

In this section we will use the notion of stack (the classical reference is [13], an introduction may be found in [8]), that of cohomology with values in a stack of 2-groups (see for example [5], and [26, Section 4])

for a review), as well as that of filtered and graded stack, for which we refer to [26, Section 2].

All the definitions and results in this section hold in the formal case.

We denote by $(\cdot)^+$ the functor from the stack of algebras to that of linear stacks defined as follows (see [10] for more details): for \mathcal{A} an algebra, \mathcal{A}^+ is the linear stack associated to the separated pre-stack

$$U \mapsto \{\text{category with a single object } \bullet \text{ and } \text{End}(\bullet) = \mathcal{A}(U)\};$$

for $f: \mathcal{A} \rightarrow \mathcal{B}$ an algebra morphism, $f^+: \mathcal{A}^+ \rightarrow \mathcal{B}^+$ is the linear functor induced by that naturally defined at the level of pre-stacks. Recall that the linear Yoneda embedding identifies \mathcal{A}^+ with the full substack of right \mathcal{A} -modules whose objects are locally free right of rank one, and f^+ with the functor induced by the extension of scalars $(\cdot) \otimes_{\mathcal{A}} \mathcal{B}$. Note that for any linear functor $\Phi: \mathcal{A}^+ \rightarrow \mathcal{B}^+$ there exist an open cover $\mathcal{U} = \{U_i\}_{i \in I}$, morphisms of algebras $f_i: \mathcal{A}|_{U_i} \rightarrow \mathcal{B}|_{U_i}$ and invertible transformations $\Phi|_{U_i} \Rightarrow f_i^+$.

4.1. (\mathcal{E}, σ) -algebroids. Let Y be a complex contact manifold. It follows from Darboux's theorem and Proposition 3.3 that any quantization algebra on Y is an \mathcal{E} -algebra, *i.e.* a \mathbb{C} -algebra locally isomorphic to $i^{-1}\mathcal{E}_M$ for any Darboux chart $i: Y \supset V \rightarrow P^*M$. Though we cannot expect a globally defined \mathcal{E} -algebra on Y , we have:

Theorem 4.1 (cf. [18]). *There exists a canonical \mathbb{C} -stack \mathbf{E}_Y which is locally equivalent to $(i^{-1}\mathcal{E}_M)^+$ for any Darboux chart $i: Y \supset V \rightarrow P^*M$.*

Note that \mathbf{E}_Y is a \mathbb{C} -algebroid stack, *i.e.* it is locally non empty and locally connected by isomorphisms (see [21, 9]). Moreover, for any (locally defined) object L of \mathbf{E}_Y , the sheaf of endomorphisms $\mathcal{E}nd_{\mathbf{E}_Y}(L)$ is a (locally defined) \mathcal{E} -algebra.

The following proposition allows us to say that \mathbf{E}_Y provides a quantization of Y .

Proposition 4.2. *The \mathbb{C} -stack \mathbf{E}_Y is filtered and there is an equivalence of graded stacks*

$$\sigma: \text{Gr}(\mathbf{E}_Y) \xrightarrow{\sim} \mathcal{O}_Y^{\text{hom}+}.$$

Then, it makes sense the following

Definition 4.3. A (\mathcal{E}, σ) -algebroid on Y is a filtered \mathbb{C} -stack \mathbf{A} locally equivalent to \mathbf{E}_Y as a \mathbb{C} -stack and endowed with an equivalence of

graded stacks

$$\nu: \text{Gr}(\mathbf{A}) \xrightarrow{\sim} \mathcal{O}_Y^{\text{hom}+}.$$

A functor of (\mathcal{E}, σ) -algebroids $(\mathbf{A}_1, \nu_1) \rightarrow (\mathbf{A}_2, \nu_2)$ is a pair (Φ, β) , where $\Phi: \mathbf{A}_1 \rightarrow \mathbf{A}_2$ is a filtered functor of \mathbb{C} -stacks and $\beta: \nu_2 \circ \text{Gr}(\Phi) \Rightarrow \nu_1$ is a graded invertible transformations of functors.

Note that any (\mathcal{E}, σ) -algebroid is locally equivalent to $(i^{-1}\mathcal{E}_M)^+$ for any Darboux chart $i: Y \supset V \rightarrow P^*M$. In particular, if \mathcal{A} is an \mathcal{E} -algebra, then \mathcal{A}^+ is a (\mathcal{E}, σ) -algebroid.

Let us briefly recall how to describe the (\mathcal{E}, σ) -algebroids by means of non-abelian cocycles. For any Darboux chart $i: Y \supset V \rightarrow P^*M$, set $\mathcal{E}_V = i^{-1}\mathcal{E}_M$. Let \mathbf{A} be a (\mathcal{E}, σ) -algebroid. By definition, there exists an open cover $X = \bigcup_{i \in I} V_i$ by Darboux charts such that $\mathbf{A}|_{V_i} \approx \mathcal{E}_{V_i}^+$. Let $\Phi_i: \mathbf{A}|_{V_i} \rightarrow \mathcal{E}_{V_i}^+$ and $\Psi_i: \mathcal{E}_{V_i}^+ \rightarrow \mathbf{A}|_{V_i}$ be \mathbb{C} -equivalences quasi-inverse one to each other. On $V_{ij} = V_i \cap V_j$ there are equivalences $\Phi_{ij} = \Phi_i \circ \Psi_j: \mathcal{E}_{V_{ij}}^+ \rightarrow \mathcal{E}_{V_{ij}}^+$, and on V_{ijk} there are invertible transformations $\alpha_{ijk}: \Phi_{ij} \circ \Phi_{jk} \Rightarrow \Phi_{ik}$ such that on V_{ijkl} the following diagram commutes

$$(4.1) \quad \begin{array}{ccc} \Phi_{ij} \circ \Phi_{jk} \circ \Phi_{kl} & \xrightarrow{\alpha_{ijk} \bullet \text{id}_{\Phi_{kl}}} & \Phi_{ik} \circ \Phi_{kl} \\ \downarrow \text{id}_{\Phi_{ij}} \bullet \alpha_{jkl} & & \downarrow \alpha_{ikl} \\ \Phi_{ij} \circ \Phi_{jl} & \xrightarrow{\alpha_{ijl}} & \Phi_{il}. \end{array}$$

(Here we denote by \bullet the horizontal composition of transformations.) Up to shrinking the open cover, one may suppose that Φ_{ij} is isomorphic to φ_{ij}^+ , for a \mathbb{C} -algebra isomorphism $\varphi_{ij}: \mathcal{E}_{V_j} \rightarrow \mathcal{E}_{V_i}$ on V_{ij} . On V_{ijk} the invertible transformation $\varphi_{ij}^+ \circ \varphi_{jk}^+ \Rightarrow \varphi_{ik}^+$ is thus given by an invertible operator $P_{ijk} \in \mathcal{E}_{V_{ijk}}$ of order m_{ijk} satisfying

$$\varphi_{ij} \circ \varphi_{jk} = \text{ad}(P_{ijk}) \circ \varphi_{ik}.$$

Finally, on V_{ijkl} the diagram (4.1) corresponds to the equality

$$P_{ijk}P_{ikl} = \varphi_{ij}(P_{jkl})P_{ijl}.$$

Recall from [26, Proposition 2.5] that the algebroid $\text{Gr}(\mathbf{A})$ is described by the 2-cocycle $\{\sigma_{m_{ijk}}(P_{ijk})\}$ with values in $\mathcal{O}_Y^{\text{hom}}$, where $\sigma_{m_{ijk}}$ denotes the symbol map $F_{m_{ijk}}\mathcal{E}_{V_i} \rightarrow \mathcal{L}^{\otimes m_{ijk}}|_{V_i}$. Since $\text{Gr}(\mathbf{A}) \approx \mathcal{O}_Y^{\text{hom}+}$, up to modify the φ_{ij} and the P_{ijk} by a coboundary, we may suppose that the P_{ijk} are of order 0 with $\sigma_0(P_{ijk}) = 1$. The non-abelian cocycle

$$(\{\mathcal{E}_{V_i}\}, \{\varphi_{ij}\}, \{P_{ijk}\})$$

is enough to reconstruct the (\mathcal{E}, σ) -algebroid \mathbf{A} (up to equivalence).

Notation 4.4. We denote by

$$\{(\mathcal{E}, \sigma)\text{-algebroids}\}_Y$$

the set of equivalence classes of (\mathcal{E}, σ) -algebroids on Y , pointed by the class of \mathbf{E}_Y .

4.2. Classification. Let $\mathbf{Aut}_{\mathcal{E}\text{-alg}}(\mathbf{E}_Y)$ denote the stack of 2-groups of (\mathcal{E}, σ) -algebroid autoequivalences of \mathbf{E}_Y . More precisely, its objects are pair (Φ, β) , where $\Phi: \mathbf{E}_Y \rightarrow \mathbf{E}_Y$ is a filtered equivalence of \mathbb{C} -stacks and $\beta: \sigma \circ \mathbf{Gr}(\Phi) \Rightarrow \sigma$ is a graded invertible transformations of functors. A morphism $\alpha: (\Phi_1, \beta_1) \Rightarrow (\Phi_2, \beta_2)$ is a filtered invertible transformation of functors $\alpha: \Phi_1 \Rightarrow \Phi_2$ making the following diagram commutative

$$(4.2) \quad \begin{array}{ccc} \sigma \circ \mathbf{Gr}(\Phi_1) & \xRightarrow{\beta_1} & \sigma \\ \text{id}_\sigma \circ \mathbf{Gr}(\alpha) \Downarrow & \nearrow \beta_2 & \\ \sigma \circ \mathbf{Gr}(\Phi_2) & & \end{array}$$

Recall that one may define the first cohomology pointed set with values in a stack of 2-groups. By definition of (\mathcal{E}, σ) -algebroid, one easily gets an isomorphism of pointed sets

$$(4.3) \quad \{(\mathcal{E}, \sigma)\text{-algebroids}\}_Y \simeq H^1(Y; \mathbf{Aut}_{\mathcal{E}\text{-alg}}(\mathbf{E}_Y)).$$

Theorem 4.5. *There is an isomorphism of pointed sets*

$$\{(\mathcal{E}, \sigma)\text{-algebroids}\}_Y \simeq H^1(Y; \Omega_Y^{1,cl}(0)).$$

Proof. In order to prove the Theorem, we will show that there is an equivalence of stacks of 2-groups on Y

$$\mathbf{Aut}_{\mathcal{E}\text{-alg}}(\mathbf{E}_Y) \approx \Omega_Y^{1,cl}(0),$$

where the sheaf $\Omega_Y^{1,cl}(0)$ is considered as a discrete stack. Since equivalent stacks of 2-groups have isomorphic cohomologies, the result will follow from (4.3).

First step. Let us define a functor of stacks of 2-groups

$$\mathbf{s}: \mathbf{Aut}_{\mathcal{E}\text{-alg}}(\mathbf{E}_Y) \rightarrow \Omega_Y^{1,cl}(0).$$

Take $(\Phi, \beta) \in \mathbf{Aut}_{\mathcal{E}\text{-alg}}(\mathbf{E}_Y)$ and choose an object L of \mathbf{E}_Y . The isomorphism $\beta_L: \sigma(\Phi(L)) \xrightarrow{\sim} \sigma(L)$ allows us to define

$$F_m \mathcal{H}om_{\mathbf{E}_Y}(L, L) \ni p \mapsto \sigma_{m-1}(\Phi(p) - p) \in \mathcal{G}r_{m-1} \mathcal{H}om_{\mathcal{O}_Y^{hom}}(\sigma(L), \sigma(L)).$$

This extends to a \mathbb{C} -derivation of degree -1 of $\mathcal{H}om_{\mathcal{O}_Y^{hom+}}(\sigma(L), \sigma(L)) \simeq \mathcal{O}_Y^{hom}$ which preserves the Poisson structure. By using (1.2), we get a section $\{\mathbf{s}(\Phi, \beta)\}_L \in \Omega_{P^*M}^{1,cl}(0)$ which depends only on the isomorphism class of L . Since any two objects M, N of \mathbf{E}_Y are locally isomorphic, it follows that $\{\mathbf{s}(\Phi, \beta)\}_M = \{\mathbf{s}(\Phi, \beta)\}_N$ locally, hence globally. This defines $\mathbf{s}(\Phi, \beta)$, which depends only on the isomorphism class of (Φ, β) .

Second step. At any point $y \in Y$, we may find a Darboux chart $i: Y \supset V \rightarrow P^*M$ containing y such that $\mathbf{E}_Y|_V \approx \mathcal{E}_V^+ = (i^{-1}\mathcal{E}_M)^+$. Let $(\Phi, \beta) \in \mathbf{Aut}_{\mathcal{E}-alg}(\mathcal{E}_V^+)$. Up to take a smaller open subset, one may suppose that σ is isomorphic to σ^+ and Φ to φ^+ for a \mathbb{C} -algebra isomorphism $\varphi: \mathcal{E}_V \rightarrow \mathcal{E}_V$. The graded transformation β is thus given by a nowhere vanishing function $b \in \mathcal{O}_{P^*M}|_V \simeq \mathcal{G}r_0(\mathcal{E}_V)$.

Locally there exists an invertible operator $Q \in F_0\mathcal{E}_V$ such that $b\sigma_0(Q) = 1$. Set $\tilde{\varphi} = \text{ad}(Q^{-1}) \circ \varphi$. Then Q defines a morphism $(\tilde{\varphi}^+, \text{id}) \Rightarrow (\Phi, \beta)$.

It follows that the functor of stacks of 2-groups

$$\left[\mathcal{E}_{V,1} \xrightarrow{\text{ad}} \mathcal{A}ut_{\mathcal{E}-alg}(\mathcal{E}_V) \right] \longrightarrow \mathbf{Aut}_{\mathcal{E}-alg}(\mathcal{E}_V^+) \quad \psi \mapsto (\psi^+, \text{id})$$

is essentially surjective, where the left-hand side denotes the stack of 2-groups associated to the crossed module $\mathcal{E}_{V,1} \xrightarrow{\text{ad}} \mathcal{A}ut_{\mathcal{E}}(\mathcal{E}_V)$ (see *loc. cit.*).

Let ψ_1, ψ_2 be two sections of $\mathcal{A}ut_{\mathcal{E}-alg}(\mathcal{E}_V)$ and consider a morphism $\alpha: (\psi_1^+, \text{id}) \Rightarrow (\psi_2^+, \text{id})$ in $\mathbf{Aut}_{\mathcal{E}}(\mathcal{E}_V^+)$. Then α is locally given by an invertible operator $R \in F_0\mathcal{E}_V$ satisfying

$$\psi_2 = \text{ad}(R) \circ \psi_1,$$

where the diagram (4.2) correspond to the equality $\sigma_0(R) = 1$. Therefore R defines a morphism $\psi_1 \rightarrow \psi_2$ in $\left[\mathcal{E}_{V,1} \xrightarrow{\text{ad}} \mathcal{A}ut_{\mathcal{E}-alg}(\mathcal{E}_V) \right]$. The above functor is thus full. Since it also faithful, we get an equivalence

$$\left[\mathcal{E}_{V,1} \xrightarrow{\text{ad}} \mathcal{A}ut_{\mathcal{E}-alg}(\mathcal{E}_V) \right] \approx \mathbf{Aut}_{\mathcal{E}-alg}(\mathcal{E}_V^+).$$

Third step. Through the above equivalence, the functor

$$\mathbf{s}: \mathbf{Aut}_{\mathcal{E}}(\mathbf{E}_Y)|_V \rightarrow \gamma_*\Omega_{\gamma^{-1}(V)}^{1,cl}(0)$$

reduces to the functor

$$\left[\mathcal{E}_{V,1} \xrightarrow{\text{ad}} \mathcal{A}ut_{\mathcal{E}-alg}(\mathcal{E}_V) \right] \rightarrow \gamma_*\Omega_{\gamma^{-1}(V)}^{1,cl}(0)$$

induced by the morphism (3.6). By Lemma 3.6 and [26, Proposition 4.4] it follows that this is an equivalence of stacks of 2-groups. The functor \mathbf{s} is thus locally, hence globally, an equivalence. \square

Corollary 4.6. *Suppose that Y is exact. Then*

$$\{(\mathcal{E}, \sigma)\text{-algebroids}\}_Y \simeq H^1(Y; \Omega_Y^{1,cl}) \times H^1(Y; \mathbb{C}_Y).$$

If moreover $H^i(Y; \mathcal{O}_Y) = 0$ for $i = 1, 2$, then

$$\{(\mathcal{E}, \sigma)\text{-algebroids}\}_Y \simeq H^2(Y; \mathbb{C}_Y) \times H^1(Y; \mathbb{C}_Y).$$

Proof. If Y is exact, then the associated contact line bundle \mathcal{L} is trivial and the Atiyah class $a(\mathcal{L})$ vanishes. Hence (1.4) splits and then one uses the Theorem 4.5. The second part of the statement follows by taking the long exact sequence associated to (1.3). \square

Corollary 4.7. *Let $Y = P^*M$ with $\dim M \geq 2$. Then*

$$\{(\mathcal{E}, \sigma)\text{-algebroids}\}_{P^*M} \simeq H^1(M; \Omega_M^{1,cl}).$$

Proof. This follows from the fact that the morphism

$$H^1(M; \Omega_M^{1,cl}) \xrightarrow{\pi^*} H^1(P^*M; \Omega_{P^*M}^{1,cl}(0))$$

is an isomorphism when $\dim M \geq 2$ (see [3, Proposition 6 (i)]). \square

It remains to consider the 1-dimensional case (cf. [3, Section 3.5]).

Let Y be a connected Riemann surface. Recall that $Y \simeq P^*Y$ has a canonical contact structure, whose contact line bundle is the sheaf Θ_Y of vector fields.

If Y is open, then $H^2(Y; \mathbb{C}) = 0$ and T^*Y is trivial. The contact structure is thus exact, and it follows from the second part of Corollary 4.6 that the (\mathcal{E}, σ) -algebroids are classified by $H^1(Y; \mathbb{C})$.

Suppose now that Y is compact of genus g . Then $H^1(Y; \Omega_Y) \simeq \mathbb{C}$ by the residue map.

If $g = 1$, then Y is a 1-dimensional complex torus and T^*Y is trivial. Again, from Corollary 4.6 it follows that the (\mathcal{E}, σ) -algebroids are classified by

$$H^1(Y; \Omega_Y) \times H^1(Y; \mathbb{C}) \simeq \mathbb{C}^3.$$

If $g \neq 1$, then the (\mathcal{E}, σ) -algebroids are classified by

$$H^1(Y; \Omega_Y(0)) \xrightarrow{\sim} H^1(Y; \mathbb{C}) \simeq \mathbb{C}^{2g}$$

(here the first isomorphism is obtained from (1.4)). In particular, up to isomorphism, \mathcal{E}_Y^+ is the unique (\mathcal{E}, σ) -algebroid on \mathbb{P}^1 (cf. Example 4.12).

Remark 4.8. Using similar techniques, one may show that Theorems 4.1 and 4.5 hold also in the framework of C^∞ manifolds, hence replacing \mathcal{E} -algebras with Toeplitz algebras (see [4]). However, in that case any quantization algebroid comes from a unique, up to isomorphism, Toeplitz algebra (cf. [11]). It follows that in the C^∞ case there is a one-to-one correspondence between quantization algebras and quantization algebroids. Note also that in the C^∞ case the classification of these objects is always given by $H^2(Y; \mathbb{C}) \times H^1(Y; \mathbb{C})$, since the exact sequence (1.4) splits and $H^1(Y; \Omega_Y^{1,cl}) \xrightarrow{\sim} H^2(Y; \mathbb{C})$.

4.3. (\mathcal{E}, σ) -algebroids on compact Kähler manifolds. Let (Y, α) be a complex contact manifold of dimension $2n + 1$, with associated contact line bundle \mathcal{L} . Consider the long exact sequence in cohomology associated to (1.4)

$$(4.4) \quad H^0(Y; \mathbb{C}_Y) \xrightarrow{\delta} H^1(Y; \Omega_Y^{1,cl}) \xrightarrow{e} H^1(Y; \Omega_Y^{1,cl}(0)) \rightarrow H^1(Y; \mathbb{C}_Y).$$

The coboundary map δ sends $\lambda \in H^0(Y; \mathbb{C}_Y)$ to $\lambda a(\mathcal{L}) = [\mathcal{D}_{\mathcal{L}^{\otimes \lambda}}]$. In particular, from the isomorphism $\Omega_Y \simeq \mathcal{L}^{\otimes -(n+1)}$ it follows that the class of the TDO-ring $\mathcal{D}_{\Omega_Y^{\otimes 1/2}}$ is sent through e to the class of the (\mathcal{E}, σ) -algebroid E_Y . Note that, up to isomorphism/equivalence, they are the unique ones which carry an anti-involution (see [25]).

Recall that, if Y is compact, then $H^i(Y; \mathbb{C}_Y)$ is finite dimensional for any i and one sets $b_i(Y) = \dim H^i(Y; \mathbb{C}_Y)$. Given a finite rank vector bundle \mathcal{V} on Y , let $c_1(\mathcal{V}) \in H^2(Y; \mathbb{C}_Y)$ be its first Chern class. Then, $c_1(\mathcal{L}) = \frac{1}{n+1} c_1(\Theta_Y)$.

Lemma 4.9 (cf. [29, Lemma 1]). *If Y is compact Kähler⁵ of $\dim Y \geq 3$, then $c_1(\mathcal{L}) \neq 0$. In particular, Y cannot be exact.*

Proposition 4.10. *Let Y be compact Kähler of $\dim Y \geq 3$ with $b_1(Y) = 0$. Then*

$$\{(\mathcal{E}, \sigma)\text{-algebroids}\}_Y \simeq H^1(Y; \Omega_Y^{1,cl}) / H^0(Y; \mathbb{C}_Y).$$

Proof. Recall that the Chern map $c_1: H^1(Y; \mathcal{O}_Y^\times) \rightarrow H^2(Y; \mathbb{C}_Y)$ factors through $H^1(Y; \Omega_Y^{1,cl})$. By Lemma 4.9, one gets $a(\mathcal{L}) \neq 0$ in $H^1(Y; \Omega_Y^{1,cl})$, so that the map δ in (4.4) is non-trivial, hence injective. One then uses the Theorem 4.5. \square

⁵By [14, Theorem 2.5], this result still holds by replacing "Kähler" by "bimeromorphically equivalent to a Kähler manifold".

Note that, for Y compact Kähler of $\dim Y \geq 3$, one also has

$$\Gamma(Y; \operatorname{Aut}_{\mathcal{E}\text{-alg}}(\mathcal{E}_Y)) \xrightarrow{\sim} H^0(Y; \Omega_Y^{1,cl}(0)) \xleftarrow{\sim} H^0(Y; \Omega_Y^{1,cl}) \simeq H^{1,0}(Y),$$

where the last isomorphism follows from the fact that every global holomorphic form on Y is closed.

For Fano manifolds the previous classification becomes very simple. Recall that a Fano manifold is a compact complex manifold such that the dual of the canonical line bundle is ample.

Proposition 4.11. *Let Y be connected and Fano of $\dim Y \geq 3$.*

(i) *If $b_2(Y) \geq 2$, then*

$$\{(\mathcal{E}, \sigma)\text{-algebroids}\}_Y \simeq \mathbb{C}.$$

(ii) *If $b_2(Y) = 1$, then, up to isomorphism, there exists a unique (\mathcal{E}, σ) -algebroid on Y .*

Proof. Since Y is compact and connected, one has $H^0(Y; \mathcal{O}_Y) \simeq \mathbb{C}$. By classical results on Fano manifolds $H^i(Y; \mathcal{O}_Y) = 0$ for $i \geq 1$, hence, using the long exact sequence in cohomology obtained from (1.3), one gets $H^i(Y; \Omega_Y^{1,cl}) \simeq H^{i+1}(Y; \mathbb{C}_Y)$ for $i \geq 0$. In particular $H^0(Y; \Omega_Y^{1,cl}) \simeq H^1(Y; \mathbb{C}_Y) = 0$, since Y is simply connected.

Being projective, Fano manifolds are Kähler. By Proposition 4.10, one gets an isomorphism

$$\{(\mathcal{E}, \sigma)\text{-algebroids}\}_Y \simeq \mathbb{C}^{b_2(Y)-1}.$$

Then (ii) follows. If $b_2(Y) \geq 2$, then Y is contact isomorphic to $P^*\mathbb{P}$ for some complex projective space \mathbb{P} (see [23, Corollary 4.2]). Since $b_2(P^*M) = b_2(M) + 1$ by the Leray-Hirsch theorem and $b_2(\mathbb{P}) = 1$, then (i) follows. \square

Example 4.12. The complex projective space \mathbb{P}^{2n+1} is a contact Fano manifold with $b_2(\mathbb{P}^{2n+1}) = 1$. By Proposition 4.11, if $n \geq 1$ there exists a unique (\mathcal{E}, σ) -algebroid on \mathbb{P}^{2n+1} . Note that, conjecturally, any contact Fano manifold with $b_2(Y) = 1$ is homogeneous (see [1] for more details).

By a result of Demailly [12], any complex contact manifold Y which is compact Kähler with $b_2(Y) = 1$ is Fano. Hence:

Corollary 4.13. *Let Y be compact Kähler of $\dim Y \geq 3$ with $b_2(Y) = 1$. Then, up to isomorphism, there exists a unique (\mathcal{E}, σ) -algebroid on Y .*

If instead $b_2(Y) \geq 2$, with Y projective of $\dim Y \geq 3$, we may use the following result to reduce the classification of (\mathcal{E}, σ) -algebroids to Corollary 4.7.

Theorem 4.14 (cf. [20, 12]). *If Y is projective with $b_2(Y) \geq 2$, then it is contact isomorphic to P^*M for some projective manifold M .*

In fact, according to [14], the same result holds replacing "projective" with "compact Kähler threefold".

5. \mathcal{E} -ALGEBRAS VS. (\mathcal{E}, σ) -ALGEBROIDS

Let (Y, α) be a complex contact manifold of dimension $2n + 1$, with associated contact line bundle \mathcal{L} .

Recall that the functor $(\cdot)^+$ from algebras to linear stacks defines a map

$$(5.1) \quad (\cdot)^+ : \{\mathcal{E}\text{-algebras}\}_Y \rightarrow \{(\mathcal{E}, \sigma)\text{-algebroids}\}_Y,$$

where the left-hand term is the (possibly empty) set of isomorphism classes of \mathcal{E} -algebras on Y .

Let \mathbf{A} be a (\mathcal{E}, σ) -algebroid. Given a global object $L \in \mathbf{A}(Y)$, the sheaf $\mathcal{E}nd_{\mathbf{A}}(L)$ of its endomorphisms is an \mathcal{E} -algebra, and the fully faithful functor $\mathcal{E}nd_{\mathbf{A}}(L)^+ \rightarrow \mathbf{A}$ is an equivalence of (\mathcal{E}, σ) -algebroids. It follows that (5.1) is surjective if and only if any (\mathcal{E}, σ) -algebroid has a global object. In this situation, we may make the set $\{\mathcal{E}\text{-algebras}\}_Y$ pointed by the class of an \mathcal{E} -algebra \mathcal{A} satisfying $\mathcal{A}^+ \approx \mathbf{E}_Y$. Note that, by [25, Corollary 3.4], we may always choose \mathcal{A} endowed with an anti-involution.

Proposition 5.1. *Let $Y = P^*M$ for M a complex manifold. If $\dim M \geq 2$, then (5.1) is surjective.*

Proof. By (3.2) and by the definition of the functor \mathbf{s} in the proof of Theorem 4.5, the morphism (5.1) identifies with the morphism

$$H^1(P^*M; \mathcal{A}ut_{\mathcal{E}\text{-alg}}(\mathcal{E}_M)) \xrightarrow{\mathbf{s}} H^1(P^*M; \Omega_{P^*M}^{1,cl}(0))$$

induced by (3.6). The result then follows by the commutative diagram (3.7) for $i = 1$, since the diagonal arrow is an isomorphism. \square

Combining the above result with Theorem 4.14, we get

Corollary 5.2. *Let Y be projective of $\dim Y \geq 3$ (or a compact Kähler threefold) with $b_2(Y) \geq 2$. Then, for any (\mathcal{E}, σ) -algebroid \mathbf{A} , there exists an \mathcal{E} -algebra \mathcal{A} such that $\mathbf{A} \approx \mathcal{A}^+$.*

Even when they exists, the classification of \mathcal{E} -algebras in general differs from that of (\mathcal{E}, σ) -algebroids. In the formal case (*i.e.* dropping the growth condition (2.1)), the following criterion⁶ gives conditions for the existence of $\widehat{\mathcal{E}}$ -algebras (*i.e.* formal \mathcal{E} -algebras) and to ensure that their classification coincides with that of $(\widehat{\mathcal{E}}, \sigma)$ -algebroids (*i.e.* formal (\mathcal{E}, σ) -algebroids).

Theorem 5.3. (i) *If $H^2(Y; \mathcal{L}^{\otimes k}) = 0$ for any $k \leq -1$, then for any $(\widehat{\mathcal{E}}, \sigma)$ -algebroid \mathbf{A} there exists an $\widehat{\mathcal{E}}$ -algebra \mathcal{A} such that $\mathbf{A} \approx \mathcal{A}^+$.*

(ii) *If moreover $H^1(Y; \mathcal{L}^{\otimes k}) = 0$ for any $k \leq -1$, there is an isomorphism of pointed sets*

$$(\cdot)^+: \{\widehat{\mathcal{E}}\text{-algebras}\}_Y \xrightarrow{\sim} \{(\widehat{\mathcal{E}}, \sigma)\text{-algebroids}\}_Y.$$

If Y is exact, then the contact line bundle \mathcal{L} is trivial, so that the conditions (i) and (ii) in Theorem 5.3 read as $H^2(Y; \mathcal{O}_Y) = 0$ and $H^1(Y; \mathcal{O}_Y) = 0$, respectively, and one may thus use Corollary 4.6. We refer to [24, 21, 30, 15, 7] for similar hypothesis in the framework of complex/algebraic symplectic/Poisson manifolds.

proof of the Theorem. First, let $Y = P^*M$ and note that there is a well defined surjective morphism of groups

$$\sigma'_{-1}: \mathcal{E}_{M,1} \rightarrow \mathcal{L}^{\otimes -1}, \quad P \mapsto \sigma_{-1}(P - 1).$$

If moreover $\sigma'_{-1}(P) = 0$, one may define $\sigma'_{-2}(P)$ and so on.

Let \mathbf{A} be a $(\widehat{\mathcal{E}}, \sigma)$ -algebroid on Y . Recall that it is described by a non-abelian cocyle $(\{\widehat{\mathcal{E}}_{V_i}\}, \{\varphi_{ij}\}, \{P_{ijk}\})$, where $\mathcal{U} = \{V_i\}_{i \in I}$ is an open cover of Y by Darboux charts, $\varphi_{ij}: \widehat{\mathcal{E}}_{V_j} \rightarrow \widehat{\mathcal{E}}_{V_i}$ are isomorphisms of \mathbb{C} -algebras on V_{ij} and $P_{ijk} \in \Gamma(U_{ijk}; \widehat{\mathcal{E}}_{V_i}(0))$ are invertible operators with $\sigma_0(P_{ijk}) = 1$, satisfying

$$(5.2) \quad \begin{cases} \varphi_{ij} \circ \varphi_{jk} = \text{ad}(P_{ijk}) \circ \varphi_{ik}, \\ P_{ijk} P_{ikl} = \varphi_{ij}(P_{jkl}) P_{ijl}. \end{cases}$$

Hence we get a 2-cocycle $\{\sigma'_{-1}(P_{ijk})\}$ with values in $\mathcal{L}^{\otimes -1}$.

By hypothesis $H^2(Y; \mathcal{L}^{\otimes -1}) = 0$, hence, up to shrink the cover, there are invertible operators $Q_{ij} \in \Gamma(U_{ij}; \widehat{\mathcal{E}}_{V_i}(0))$ with $\sigma_0(Q_{ij}) = 1$ satisfying

⁶Conjecturally, this criterion works also in the holomorphic case.

$\sigma'_{-1}(P_{ijk}) = \sigma'_{-1}(Q_{ij}) + \sigma'_{-1}(Q_{jk}) - \sigma'_{-1}(Q_{ik})$. Set

$$(5.3) \quad \begin{cases} \varphi_{ij}^1 = \text{ad}(Q_{ij}^{-1}) \circ \varphi_{ij}, \\ P_{ijk}^1 = \varphi_{ij}^1(Q_{jk}^{-1})Q_{ij}^{-1}P_{ijk}Q_{ik}. \end{cases}$$

This gives a non-abelian cocycle $(\{\widehat{\mathcal{E}}_{V_i}\}, \{\varphi_{ij}^1\}, \{P_{ijk}^1\})$ equivalent to (5.2) and satisfying $\sigma_0(P_{ijk}^1) = \sigma'_{-1}(P_{ijk}^1) = 0$. We may thus define the 2-cocycle $\{\sigma'_{-2}(P_{ijk}^1)\}$ with values in $\mathcal{L}^{\otimes -2}$ and so on. The non abelian cocycle (5.2) is finally equivalent to a non abelian cocycle of the form $(\{\widehat{\mathcal{E}}_{V_i}\}, \{\phi_{ij}\}, \{0\})$, which represent an $\widehat{\mathcal{E}}$ -algebra \mathcal{A} such that $\mathcal{A}^+ \approx \mathbf{A}$.

(ii) is proved in a similar way. \square

Proposition 5.4. *If Y is Fano of $\dim Y \geq 3$, then*

$$\{\widehat{\mathcal{E}}\text{-algebras}\}_Y \simeq \{(\widehat{\mathcal{E}}, \sigma)\text{-algebroids}\}_Y.$$

Proof. Let Y be compact. By definition, it is Fano if the dual of $\Omega_Y \simeq \mathcal{L}^{\otimes -(n+1)}$ is ample, hence if and only if \mathcal{L} is ample. By the Kodaira vanishing theorem, one gets $H^i(Y; \mathcal{L}^{\otimes k}) = 0$ for any $k \leq -1$ and $i \leq \dim Y - 1$. The result then follows from Theorem 5.3. \square

In particular, the above isomorphism holds if Y is compact Kähler of $\dim Y \geq 3$ with $b_2(Y) = 1$, since it is Fano (see [12, Corollary 3]). If moreover Y is connected, from (the formal analogue of) Proposition 4.11 (ii) it follows that these sets are singletons.

Proposition 5.5. *Let M be a complex manifold. The morphism of pointed sets*

$$(\cdot)^+ : \{\widehat{\mathcal{E}}\text{-algebras}\}_{P^*M} \rightarrow \{\widehat{\mathcal{E}}\text{-algebroids}\}_{P^*M}$$

is surjective. If moreover $\dim M \geq 3$, then it is an isomorphism.

Proof. By (the formal analogue of) Proposition 5.1, the map $(\cdot)^+$ is surjective if $\dim M \geq 2$.

If $\dim M = 1$, then $P^*M \simeq M$ and $\mathcal{O}_M(1)$ identifies with the sheaf Θ_M of holomorphic vector fields. Since $H^2(M; \Theta_M^{\otimes k}) = 0$ for any k , by Theorem 5.3 (i) the map $(\cdot)^+$ is surjective.

Set $\dim M = n + 1$. Recall that one has

$$R\pi_* \mathcal{O}_{P^*M}(k) \simeq \begin{cases} S_{\mathcal{O}_M}^k(\Theta_M) & \text{if } k \geq 0; \\ S_{\mathcal{O}_M}^{-k-n-1}(\Omega_M^1) \otimes \Omega_M[-n] & \text{if } k \leq -1; \end{cases}$$

where $R\pi_*$ denotes the right derived direct image of the projection $\pi: P^*M \rightarrow M$, and $S_{\mathcal{O}_M}^i$ the i -th symmetric product over \mathcal{O}_M ($= 0$ for $i \leq -1$).

If $k \leq -1$, we thus have

$$H^i(P^*M; \mathcal{O}_{P^*M}(k)) \simeq H^i(M; R\pi_* \mathcal{O}_{P^*M}(k)) \simeq H^{i-n}(M; S_{\mathcal{O}_M}^{-k-n-1}(\Omega_M^1) \otimes \Omega_M).$$

It follows that $H^i(P^*M; \mathcal{O}_{P^*M}(k)) = 0$ for any $k \leq -1$ and $i \leq n-1$. If $\dim M \geq 3$, then $(\cdot)^+$ is an isomorphism by Theorem 5.3 (ii). \square

Thanks to (the formal analogue of) Corollary 4.7, one recovers that the $\widehat{\mathcal{E}}$ -algebras on P^*M are classified by $H^1(M; \Omega_M^{1,cl})$ if $\dim M \geq 3$ (cf. [3], where this is proved by showing that $H^1(P^*M; \widehat{\mathcal{E}}_{M,1}) = 0$).

By using Theorem 4.14, one gets

Corollary 5.6. *Let Y be projective of $\dim Y \geq 5$ with $b_2(Y) \geq 2$. Then*

$$\{\widehat{\mathcal{E}}\text{-algebras}\}_Y \simeq \{(\widehat{\mathcal{E}}, \sigma)\text{-algebroids}\}_Y.$$

The “exotic” phenomena, *i.e.* the existence of non isomorphic $\widehat{\mathcal{E}}$ -algebras giving rise to equivalent $(\widehat{\mathcal{E}}, \sigma)$ -algebroids, appear if $\dim Y \leq 3$. Precisely, for $Y \simeq P^*M$ with M a compact Kähler surface different from \mathbb{P}^2 (in which case Y would be Fano), and for curves. Some computations for these cases may be found in [3]. Let us briefly consider the case of curves⁷.

Let Y be a connected compact Riemann surface of genus g . Recall that the contact line bundle is Θ_Y . By Proposition 5.5, for any $(\widehat{\mathcal{E}}, \sigma)$ -algebroid \mathbf{A} there exists an $\widehat{\mathcal{E}}$ -algebra \mathcal{A} such that $\mathbf{A} \approx \mathcal{A}^+$. However, the classification of the $(\widehat{\mathcal{E}}, \sigma)$ -algebroids given at the end of Section 4 differs from that of the $\widehat{\mathcal{E}}$ -algebras, since $H^1(Y; \Theta_Y^{\otimes k})$ is in general not zero for $k \leq -1$ (in particular $H^1(Y; \Theta_Y^{\otimes -1}) \simeq \mathbb{C}$ by the residue map).

Let $g \geq 2$. Since $H^1(Y; \Theta_Y^{\otimes k}) = 0$ for any $k \leq -2$, we have $H^1(Y; \widehat{\mathcal{E}}_{Y,1}) \simeq H^1(Y; \Theta_Y^{\otimes -1}) \simeq \mathbb{C}$ and the exact sequence (3.4) reduces to

$$0 \rightarrow \mathbb{C} \rightarrow \{\widehat{\mathcal{E}}\text{-algebras on } Y\} \rightarrow \mathbb{C}^{2g} \rightarrow 0.$$

Since it splits, it follows that the $\widehat{\mathcal{E}}$ -algebras are classified by \mathbb{C}^{2g+1} .

⁷Note that the original manuscript contains an error. See <http://www.math.jussieu.fr/~boutet/ComplexStar.pdf> for a corrected version.

Let $g = 0$. Then $Y \simeq \mathbb{P}^1$. Since $H^i(\mathbb{P}^1; \Omega_{\mathbb{P}^1}(0)) = 0$ for $i = 0, 1$, by the exact sequence (3.4) it follows that the $\widehat{\mathcal{E}}$ -algebras are classified by $H^1(\mathbb{P}^1; \widehat{\mathcal{E}}_{\mathbb{P}^1,1})$. This is far to be trivial, since $H^1(\mathbb{P}^1; \Theta_{\mathbb{P}^1}^k) \simeq H^1(\mathbb{P}^1; \mathcal{O}_{\mathbb{P}^1}(2k))$ is never vanishing for $k \leq -1$.

If Y is compact Kähler of $\dim Y \geq 5$ with $b_2(Y) \geq 2$, it is conjectured that Y is contact isomorphic to P^*M for a compact Kähler manifold M . One knows that $\mathcal{L}^{\otimes -1}$ is not pseudo-effective (see [12]). This leads us to consider the following

Conjecture 5.7. *Let Y be compact Kähler of $\dim Y = 2n + 1 \geq 5$ with $b_2(Y) \geq 2$. Then $H^i(Y; \mathcal{L}^{\otimes k}) = 0$ for any $k \leq -1$ and $i \leq n - 1$.*

This would imply that any $(\widehat{\mathcal{E}}, \sigma)$ -algebroid A on such Y has a global object, and that

$$\{\widehat{\mathcal{E}}\text{-algebras}\}_Y \simeq \{(\widehat{\mathcal{E}}, \sigma)\text{-algebroids}\}_Y$$

whenever $\dim Y \geq 7$.

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DIPARTIMENTO DI MATEMATICA PURA ED APPLICATA, UNIVERSITÀ DI PADOVA,
VIA TRIESTE 63, 35121 PADOVA, ITALY
E-mail address: `pietro@math.unipd.it`